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Stimuli shift the position of traveling waves in neural fields with synaptic depression

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Summary

Scientific Question: How do brains encode and predict motion?

- The firing-rate of populations of neurons is sufficient to explain dynamics of large-scale neural networks.
- We can approximate large discrete networks using integro-differential equations called *neural field models*.
- Synaptic depression allows for biologically realistic traveling pulses in neural field models.
- External stimuli can adjust the position of traveling pulses.

Background: Synaptic Depression

- When a pre-synaptic neuron fires, it releases neurotransimiters into the synaptic cleft separating it from the post-synaptic neuron.
- When neurons fire repeatedly, they will deplete their store of neurotransmiters faster than they replenish them.
- This results in reduced stimulation of the post-synaptic neuron and a reduced firing-rate. We call this **synaptic depression**.

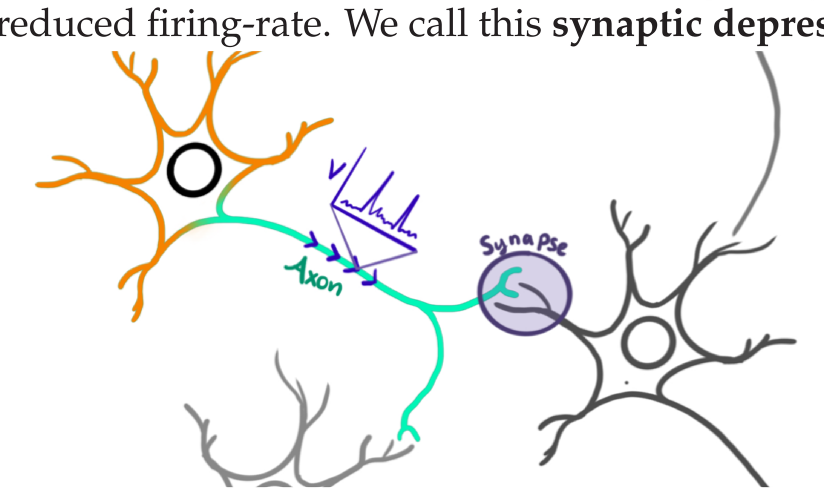


Image courtesy of Heather Cihak.

One-Dimensional Neural-Field Model

$$\tau_u \frac{\partial}{\partial t} u(x,t) = \underbrace{-u(x,t)}_{\text{decay}} + \underbrace{\int_{\mathbb{R}} w(|x-y|)\, q(y,t) f[u(y,t)]\, dy}_{\text{non-linear spatial operator}} + \underbrace{\varepsilon I_u(x,t)}_{\text{stimulus}}$$

$$\tau_q \frac{\partial}{\partial t} q(x,t) = \underbrace{1 - q(x,t)}_{\text{replenishment}} - \underbrace{\beta q(x,t) f[u(x,t)]}_{\text{depletion}} + \underbrace{\varepsilon I_q(x,t)}_{\text{stimulus}}$$

$$w(|x-y|) = \tfrac{1}{2} e^{-|x-y|} \qquad \gamma = \frac{1}{1+\beta}$$

$$f(\cdot) = H(\cdot - \theta) \qquad A(t) = \{x \in \mathbb{R} \mid u(x,t) \geq \theta\}$$

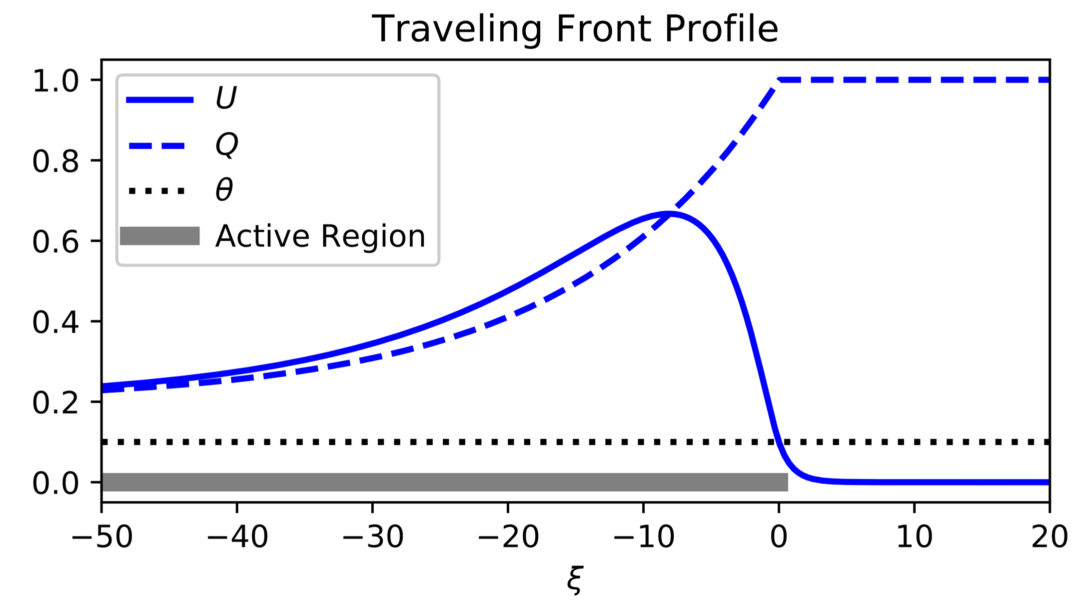
- $u(x,t)$ – neural activity (normalized firing-rate)
- $q(x,t)$ – synaptic efficacy ($q < 1$ represents synaptic depression)
- τ_u – time-scale of neural activity
- τ_q – time-scale of synaptic repleneshment
- β – time-scale (relative to τ_q) of synaptic depletion
- γ – effective synaptic time-scale (relative to τ_q)
- w – a weight kernel that encodes the network connectivity
- f – a non-linear firing-rate function
- θ – the firing-rate threshold
- $A(t)$ (active region) – the subset of domain in which neural activity is sufficient to simtulate other areas of the network
- $\varepsilon I_u, \varepsilon I_q$ – small external stimulii

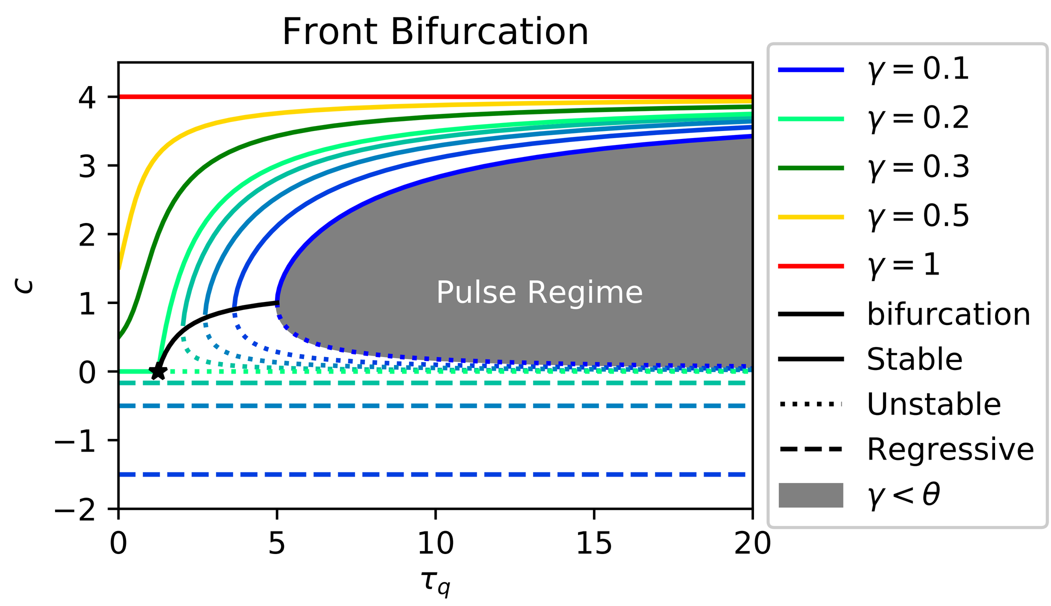
Traveling Wave Solutions

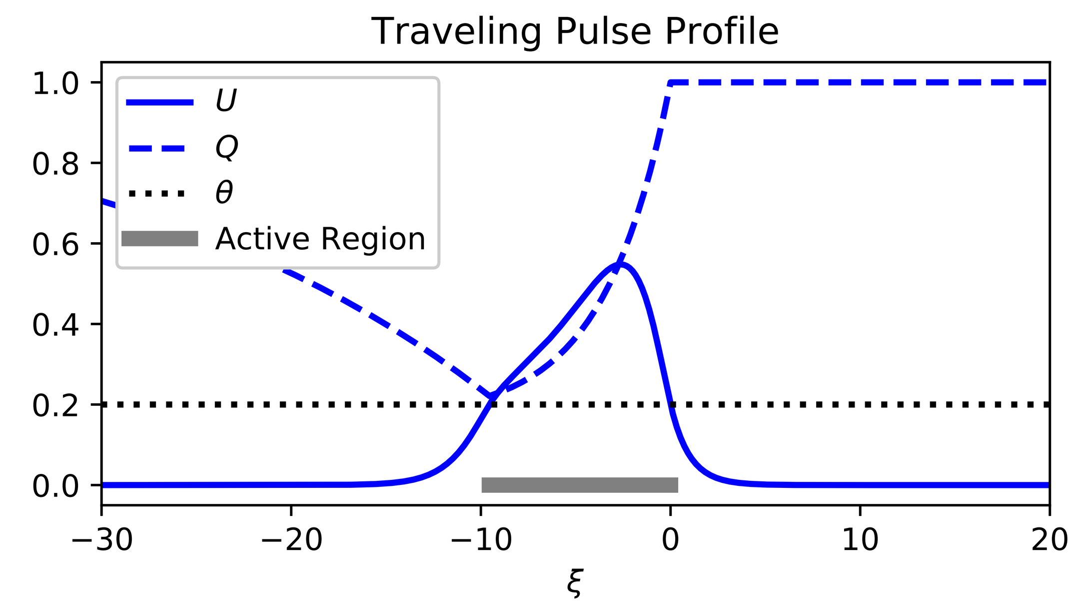
- Convert to characteristic coordinates: $\xi = x - ct$
- Traveling wave solutions $u(x,t) = U(\xi)$, $q(x,t) = Q(\xi)$ satisfy the linear system

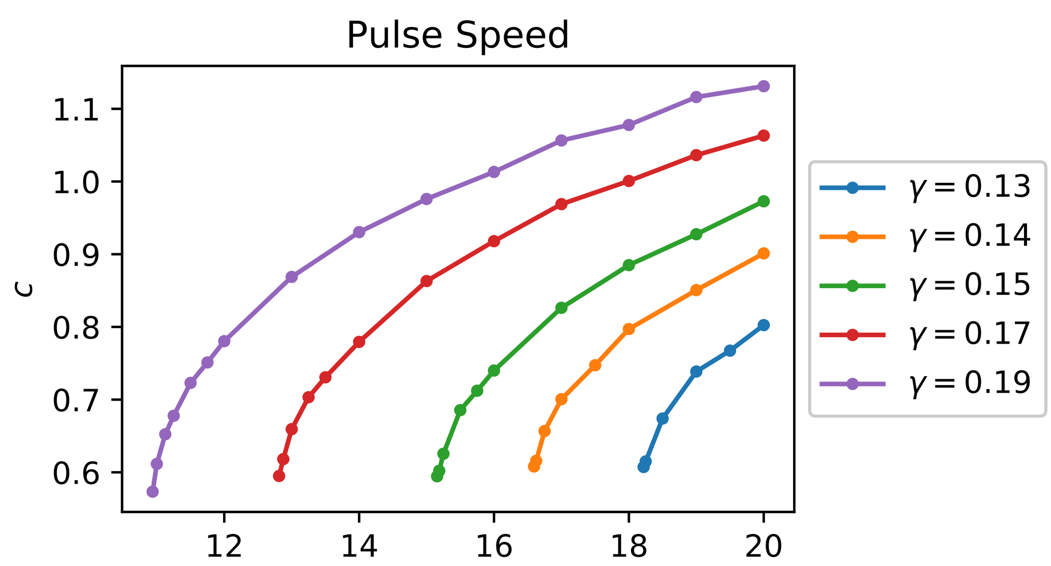
$$-c\tau_u \frac{d}{d\xi} U(\xi) = -U(\xi) + \int_A w(|\xi - y|) Q(y)\, d\xi'$$
$$-c\tau_q \frac{d}{d\xi} Q(\xi) = 1 - Q(\xi) - \beta Q(\xi) I_A(\xi)$$

- This gives a consistency equation for the speed c (and pulse width).



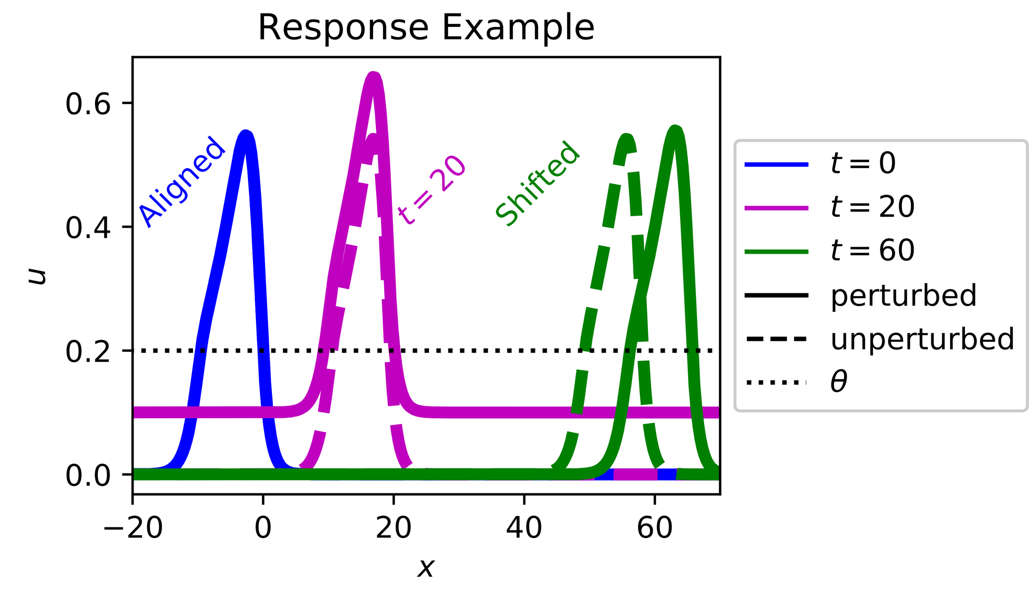






Wave Response

- These solutions have fixed speeds.
- Our visual system is capable of tracking and predicting the location of objects with a variety of speeds.
- **Can we augment the model in a biologically realisitc way to account for this variation in speed?**
- These waves are *marginally stable* – when stimulated, they tend toward a translate of the original traveling wave. Below we see snapshots for $\varepsilon I_u = 0.1\delta(t-20)$.



- The amount of translation is called the *wave response*, denoted $\nu(t)$.

Asymptotics

- Expand about the traveling wave solution

$$u(\xi,t) = U(\xi - \varepsilon\nu(t)) + \varepsilon\phi(\xi,t) + \mathcal{O}(\varepsilon^2)$$
$$q(\xi,t) = Q(\xi - \varepsilon\nu(t)) + \varepsilon\psi(\xi,t) + \mathcal{O}(\varepsilon^2)$$

- Substitute into the model, linearize, and extract the $\mathcal{O}(\varepsilon)$ equation:

$$\begin{bmatrix}\tau_u & 0\\ 0 & \tau_q\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}_t + \mathcal{L}\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right) = \underbrace{\begin{bmatrix}I_u + \tau_u U'\nu'\\ I_q + \tau_q Q'\nu'\end{bmatrix}}_{\text{RHS}}$$

where

$$\mathcal{L}\left(\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right) = \begin{bmatrix}\phi\\ \psi\end{bmatrix} - c\begin{bmatrix}\tau_u & 0\\ 0 & \tau_q\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}_\xi + \begin{bmatrix}-wQf'(U)* \cdot & -wf(U)*\cdot\\ \beta Qf'(U) & \beta f(U)\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix}$$

- We apply the **Fredholm alternative**: A unique bounded solution exists if the right-hand-side is orthogonal to the null-space of the adjoint.
- The one-dimensional null-space $(v_1, v_2) \in \mathcal{N}(\mathcal{L}^*)$ satisfies

$$-c\tau_u v_1' = v_1 - f'(U)Q\int_{\mathbb{R}} w(|y-\xi|)v_1(y)\, dy + \beta Q f'(U) v_2$$
$$-c\tau_q v_2' = v_2 - f(U)\int_{\mathbb{R}} w(|y-\xi|)v_1(y)\, dy + \beta f(U) v_2.$$

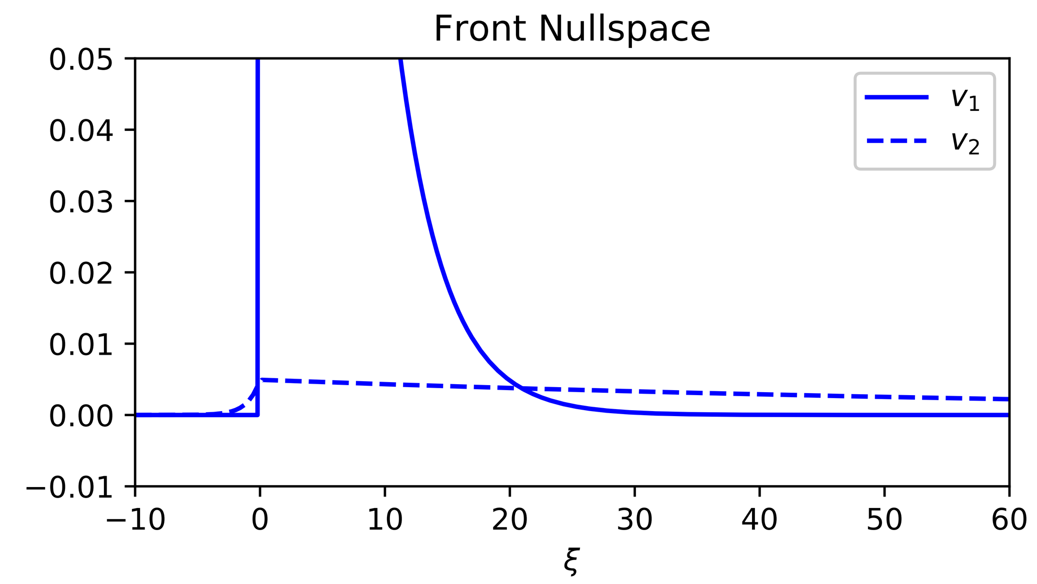
- **Asymptotic approximation**:

$$\nu(t) = -\frac{\int_{\mathbb{R}} v_1 \int_0^t I_u(\xi,\tau)\, d\tau + v_2 \int_0^t I_q(\xi,\tau)\, d\tau\, d\xi}{\int_{\mathbb{R}} \tau_u U' v_1 + \tau_q Q' v_2\, d\xi}.$$

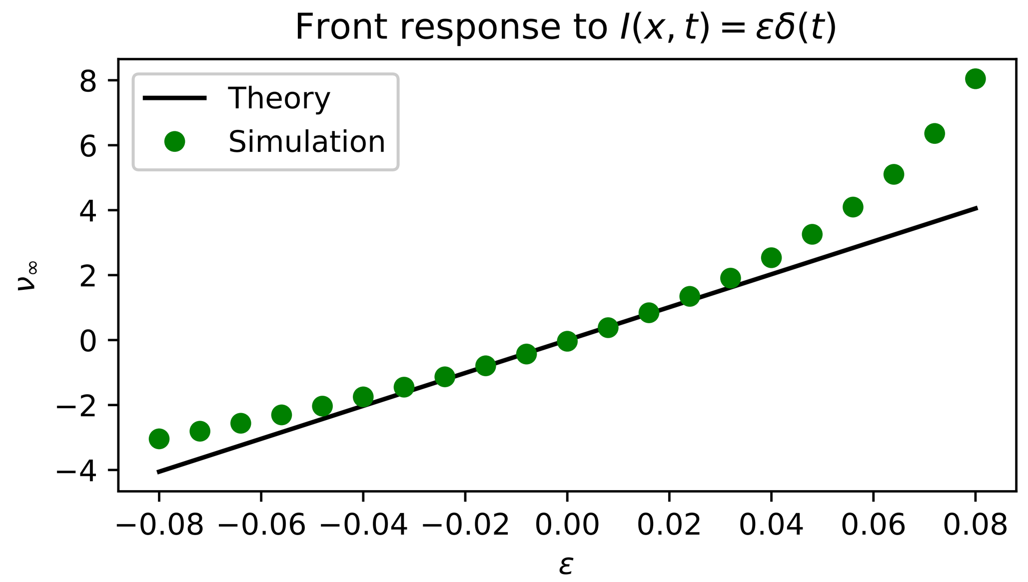
Results

- Front null-space:

$$v_1(\xi) = H(\xi)e^{-\xi/c\tau_u}$$
$$v_2(\xi) = \frac{c\tau_u}{2(1+c\tau_u)(1+\beta+c\tau_q)}\left(H(-\xi)e^{\xi} + H(\xi)e^{-\xi/c\tau_q}\right)$$

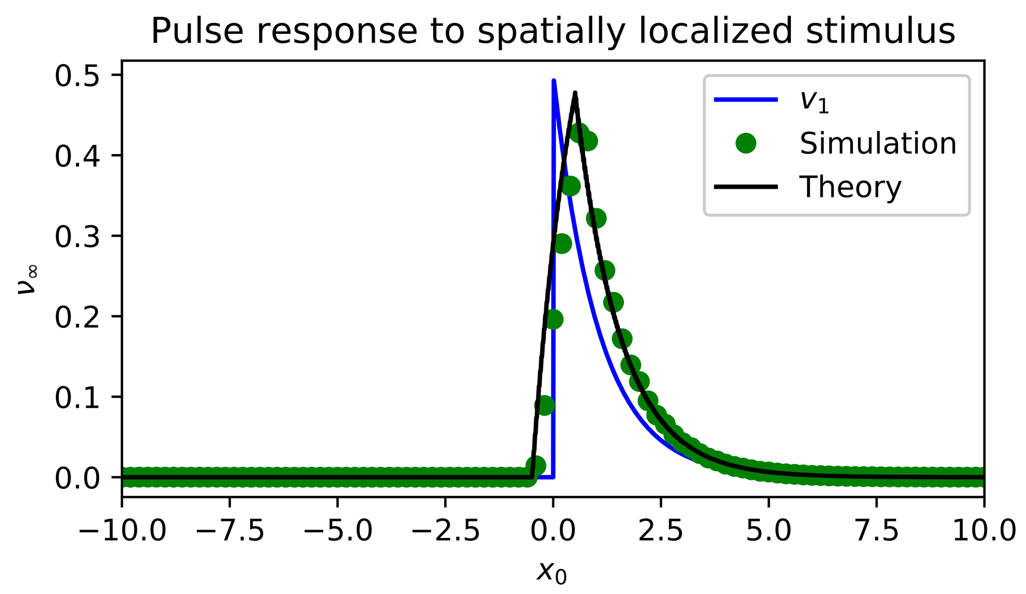


- Front response to global stimulus $\varepsilon I_u = \varepsilon\delta(t)$.



- Pulse response to unit-width square pulse centered at x_0.

$$\varepsilon I_u = \tfrac{1}{20}\delta(t) I_{(x_0 - .5, x_0 + .5)}$$



References, Funding, and Links

- Tsodyks, et al. (1998) Neural Computation
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https://shawsa.github.io/presentations/20230310_recruitment_poster.html